Lecture notes on Dynamic Meteorology for the Advanced Met. training course prepared by Dr.Somenath Dutta.

Hydrodynamic Instability

Definition: A mean flow field is said to hydro dynamically unstable if a small perturbation, introduced into the mean flow, grows spontaneously by extracting energy from the mean flow.

Classification of hydrodynamic instability:

Categorization of hydrodynamic instability may either be based on the state of the mean flow or on the mode of perturbation introduced.

Based on the former, hydro dynamic instability may be dynamic or static according as the mean flow is there or not. While discussing Barotropic instability, Baroclinic instability or Inertial instability, we always consider a mean flow having some speed. These are examples of dynamic instabilities. But while discussing Brunt Vaisala instability, we need not to take care of the mean flow. This is example of static instability.

Based on the later, hydro dynamic instability may be of two types, viz., parcel instability and wave instability. Some times perturbation may be introduced as a displacement to an air parcel and it is examined under what condition the parcel is moving away from its mean position. This is known as parcel instability. Brunt-Vaisala instability and Inertial instability are examples of parcel instability. In another case, the perturbation is given in the form of a wave super imposed on a mean flow and examined under what conditions the wave is being amplified. This is known as wave instability. Barotropic and Baroclinic instabilities are examples of wave instability.

The above categorization is shown below in a tabular form:

Hydrodynamic Instability			
Based On The State Of Mean Flow		**Based On The Mode Of Perturbation**	
Static Instability Example: Brunt Vaisala instability.	Dynamic Instability Examples: Inertial, Barotropic, Baroclinic	Parcel Instability Inertial, Brunt Vaisala	Wave Instability Barotropic, Baroclinic

Brunt Vaisala instability:

To analyse the Brunt Vaisala instability, we consider an air parcel embedded in a static mean flow. Let the parcel be displaced vertically.

If ρ_P and ρ_E are the densities of air inside the parcel and that of environmental air at new position then the net buoyancy force acting on the air parcel is $V(\rho_E - \rho_P)g$; where V is the volume of air parcel. Thus considering only the buoyancy force, the vertical momentum equation of the air parcel is

$\frac{dw}{dt} = \frac{V(\rho_E - \rho_P)g}{V\rho_P} = g\frac{\rho_E - \rho_P}{\rho_P}$ ……(1). Since pressure across the boundary of the parcel is continuous, it follows that $P = \rho_P RT_P = \rho_E RT_E$; Where T_p and T_E are the temperature of the air parcel and that of environmental air and 'P' is the pressure across the boundary of air parcel.

Hence it 'ς' denotes the vertical displacement, then we have

$$\frac{d^2\varsigma}{dt^2} = g\,\frac{T_P - T_E}{T_E} \ldots\ldots(2)$$

Now, $T_P(\varsigma) = T_P(0) + \varsigma\left(\frac{\partial T_P}{\partial z}\right)_{z=0} + \ldots.$

$$= T_P(0) - \varsigma\Gamma_P + \ldots.$$

It is assumed that a dry air parcel follows a dry adiabatic line and a moist air parcel follows a saturated (pseudo) adiabatic line. Hence Γ_P is either dry adiabatic lapse rate (DALR) or saturated adiabatic lapse rate (SALR). So we may write $\Gamma_P = \Gamma_a$; where, 'a' stands for adiabatic, dry or saturated, whatever is applicable.

Hence, $T_P(\varsigma) = T_P(0) - \varsigma\Gamma_a$ (neglecting higher order terms).

Similarly, the environmental temperature at $z = \varsigma$, is given by,

$T_E(\varsigma) = T_E(0) - \varsigma\Gamma_E$, where, Γ_E is the environmental lapse rate.

Substituting these expressions of $T_P(\varsigma)$ and $T_E(\varsigma)$ in (2), we obtain

$$\frac{d^2\varsigma}{dt^2} = -N^2\varsigma \ldots\ldots(3), \text{ where, } N^2 = g\,\frac{\Gamma_P - \Gamma_E}{T_E}.$$

The above equation has a stable sine/cosine solution of $N^2 > 0$ and has an unstable exponential solution if $N^2 < 0$.

Thus the vertical displacement of the parcel is stable if $N^2 > 0$ i.e, if the environmental lapse rate is less then the adiabatic lapse rate other wise unstable if environmental lapse rate exceeds that of parcel. N is known as Brunt Vaisala frequency.

Inertial instability: We consider an air parcel embedded in a mean zonally geostrophic flow. Suppose, the air parcel be displaced meridionally from $y = y_0$, to $y = y_0 + \delta y$ during the period $t = t_0$ and $t = t_0 + \delta t$. Then at the new position, the horizontal equation of motion can be written as,

$$\frac{du}{dt} = fv = f\,\frac{dy}{dt} \ldots\ldots(4)$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -fu + fu_g \ldots.(5).$$

Integrating (4) between initial and final position we obtain,

$$u(t_0 + \delta t) - u(t_0) = f\left[y(t_0 + \delta t) - y(t_0)\right]$$

$$\Rightarrow u(y_0 + \delta y) - u(y_0) = f\left[y_0 + \delta y - y_0\right] = f\,\delta y \ldots\ldots(6)$$

Writing the equation (5) at $y = y_0 + \delta y$, we obtain,

$$\frac{dv}{dt} = -f\left[u(y_0 + \delta y) - u_g(y_0 + \delta y)\right]$$

$$= -f\left\{[u(y_0) + f\delta y + ...] - \left[u_g(y_0) + \left(\frac{\partial u_g}{\partial y}\right)_{y=y_0}\delta y + ...\right]\right\} \text{(Using (6))}$$

At the initial position the air parcel was embedded in the meanflow, which is zonaly geostrophic. Hence, $u(y_0) = u_g(y_0)$.

Thus at $y = y_0 + \delta y$, $\frac{dv}{dt} = -f\delta y\left(f - \frac{\partial u_g}{\partial y}\right)$……(7)

Multiplying both sides of (7) by $v = \frac{d}{dt}(\delta y)$ and then integrating between initial and final position, we obtain,

$\frac{dK'}{dt} = -f\varsigma_a\frac{(\delta y)^2}{2}$…..(8), where, K' is the eddy meridional kinetic energy of the parcel and ς_a is the absolute vorticity of the mean flow. Since the RHS of (8), represents the rotational K.E of the mean flow, it appears that perturbation grows by extracting rotational K.E of the mean flow.

In the northern hemisphere f > 0. Thus the K.E of the parcel will increase with time if $\varsigma_a < 0$, i.e., if the mean flow has absolute anticyclonic vorticity and will decrease if $\varsigma_a > 0$, i.e., if the mean flow has absolute cyclonic vorticity and neutral if $\varsigma_a = 0$.

In the southern hemisphere, f < 0. Thus the K.E of the parcel will increase with time if $\varsigma_a > 0$, which corresponds to absolute anticyclonic vorticity in the southern hemisphere and will decrease if $\varsigma_a < 0$, which again corresponds to absolute cyclonic vorticity in the southern hemisphere and neutral if $\varsigma_a = 0$.

Thus, a mean flow with cyclonic vorticity is inertially stable and with anticyclonic vorticity is inertially unstable. The result may be interpreted as follows:

A mean flow with a cyclonic absolute vorticity is itself active enough so that it cannot spare its energy to grow perturbation in it, where as that with an anticyclonic absolute vorticity is not active enough, so that it can spare its energy to the perturbation to grow.

Barotropic Instability:

Definition: A zonal mean flow field is said to be barotropically unstable if a small perturbation, introduced in it, grows spontaneously by extracting kinetic energy from the mean flow.

Barotropic instability analysis:

To, analyse the barotropic instability; we start with the non divergent barotropic model. The governing equation for this is given by

$$\frac{\partial \varsigma}{\partial t} = -\vec{V}.\vec{\nabla}(\varsigma + f) = -\left(u\frac{\partial \varsigma}{\partial x} + v\frac{\partial \varsigma}{\partial y}\right) - v\beta \text{……(1)}$$

We apply have perturbation technique, following which we split the fields into basic and perturbation parts as below:

$u = \bar{u}(y) + u'(x, y, t)$

$v = 0 + v'(x, y, t)$

Hence, $\varsigma = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} - \frac{d\bar{u}}{dy} = \varsigma' + \bar{\varsigma}$

Substituting in the above governing equation, we obtain,

$$\frac{\partial \varsigma'}{\partial t} = -\bar{u}\frac{\partial \varsigma'}{\partial x} - v'\left(\beta - \frac{d^2\bar{u}}{dy^2}\right) \ldots\ldots(2)$$

Here, we introduce, perturbation stream function, $\psi'(x, y, t)$, such that,

$v' = \frac{\partial \psi'}{\partial x}$ and $u' = -\frac{\partial \psi'}{\partial y}$, so that, $\varsigma' = \nabla^2 \psi'$

Hence (2) reduces to,

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \frac{\partial \psi'}{\partial x}\left(\beta - \frac{d^2\bar{u}}{dy^2}\right) = 0 \ldots..(3)$$

We seek the wave solution for (3) like

$$\psi'(x, y, t) = A(y)e^{i(kx - \nu t)} \ldots..(4)$$

Subject to the boundary condition

$$A(\pm d) = 0 \ldots\ldots(5)$$

Substituting (4) in (3) we have

$$\left(-k^2 A + \ddot{A}\right)\left(-i\nu + ik\bar{u}\right) + ikA\left(\beta - \ddot{\bar{u}}\right) = 0 \ldots..(6)$$

Multiplying both sides of (6) by A^*, the complex conjugate of A, we obtain

$$\left[-k^2|A|^2 + \frac{d\left(A^*\dot{A}\right)}{dy} - \left|\frac{dA}{dy}\right|^2\right]\left(-\nu + \bar{u}k\right) + k|A|^2\left(\beta - \ddot{\bar{u}}\right) = 0 \ldots\ldots(7)$$

Integrating the above with respect to 'y' between $y = \pm d$, we obtain

$$\int_{-d}^{+d}\left\{k^2|A|^2 + \left|\frac{dA}{dy}\right|^2\right\}dy = \int_{-d}^{+d}\frac{|A|^2}{(\bar{u} - c)}\left(\beta - \ddot{\bar{u}}\right)dy \ldots\ldots.(8)$$

Now, $c = c_r + ic_i$, where, c_r and c_i are respectively the real and imaginary part of the phase velocity 'c'. So, $\bar{u} - c = \left(\bar{u} - c_r\right) - ic_i$. Multiplying the numerator and denominator of the integrand on RHS of (8) by the complex conjugate of $(\bar{u} - c)$, we obtain,

$$\int_{-d}^{+d}\left\{k^2|A|^2+\left|\frac{dA}{dy}\right|^2\right\}dy=\int_{-d}^{+d}\frac{|A|^2\left[(\bar{u}-c_r)+ic_i\right]}{\left[(\bar{u}-c_r)^2+c_i^{\,2}\right]}(\beta-\ddot{\bar{u}})dy$$

L.H.S of the above equation is a pure real number, hence the R.H.S has to be so, which requires

$c_i\int_{-d}^{+d}\frac{(\beta-\ddot{\bar{u}})}{(\bar{u}-c_r)^2+c_i^{\,2}}dy=0$. Since, $c_i\neq 0$, hence, $\int_{-d}^{+d}\frac{(\beta-\ddot{\bar{u}})}{(\bar{u}-c_r)^2+c_i^{\,2}}dy=0$ …..(9).

Since the denominator of the integrand in (9) is a positive definite quantity, hence, it is always positive, Thus the above definite integral to vanish, $(\beta-\ddot{\bar{u}})$ mush change sign within the limit of integration. This further requires that there must exist some point, say $y=y_c$, between $y=\pm d$, such that $(\beta-\ddot{\bar{u}})_{y=y_c}=0$ ……(10). This is the necessary condition for barotropic instability. Thus for a mean zonal flow to be barotropically unstable, the necessary condition is that at same intermediate latitude the mean flow has an extreme absolute vorticity.

Energetics of barotropic instability: To study the energetics of barotropic instability, first we will show that in the non-divergent barotropic model the mean kinetic energy remains conserved. For that we start with non divergent vorticity equation,

$$\frac{\partial\varsigma}{\partial t}=-\vec{V}.\vec{\nabla}(\varsigma+f)=-\left(u\frac{\partial\varsigma}{\partial x}+v\frac{\partial\varsigma}{\partial y}\right)-v\beta$$

$$\left(\frac{\partial}{\partial t}+\vec{V}.\vec{\nabla}\right)\nabla^2\psi+\beta\frac{\partial\psi}{\partial x}=0$$

$$\vec{\nabla}.\left(\frac{\partial}{\partial t}+\vec{V}.\vec{\nabla}\right)\vec{\nabla}\psi+\beta\frac{\partial\psi}{\partial x}=0$$

Multiplying above by 'ψ', we obtain,

$$\psi\vec{\nabla}.\left(\frac{\partial}{\partial t}+\vec{V}.\vec{\nabla}\right)\vec{\nabla}\psi+\beta\psi\frac{\partial\psi}{\partial x}=0$$

$$\vec{\nabla}.\left[\psi\left(\frac{\partial}{\partial t}+\vec{V}.\vec{\nabla}\right)\vec{\nabla}\psi\right]-\vec{\nabla}\psi.\left(\frac{\partial}{\partial t}+\vec{V}.\vec{\nabla}\right)\vec{\nabla}\psi+\beta\frac{\partial\left(\psi^2/2\right)}{\partial x}=0$$

Integrating the above over a volume 'σ', consisting of from $y=-d$ to $y=d$, from bottom to top of the atmosphere and over an entire wavelength of a barotropic wave, we obtain,

$\frac{dK}{dt}=0$, where, $\frac{d}{dt}\equiv\frac{\partial}{\partial t}+\vec{V}.\vec{\nabla}$ and $K=\iiint_{\sigma}\frac{\vec{\nabla}\psi.\vec{\nabla}\psi}{2}d\sigma$.

Now, if $K=\overline{K}+K'$, then we have,

$\frac{dK'}{dt}=-\frac{d\overline{K}}{dt}$. Thus, the barotropic instability grows by extracting K.E from the mean flow.

Baroclinic instability

For a mean flow to be baroclinic unstable, first of all the mean flow should be baroclinic, i.e., there should exist a north-south temperature gradient in the mean. Due to that atmosphere possesses a certain amount of available potential energy (APE=I.E+P.E). Now if this existing N-S temperature gradient is increased by warming the warm latitude & cooling the cold latitude, then APE will go on increasing. Once APE exceeds certain threshold value, depending on the prevailing mean flow, the westerly flow becomes baroclinic unstable. This instability is demonstrated by waves super-imposed in basic westerly flow. Wave patterns are seen in contour field, thermal field etc., as shown in the figure 1.

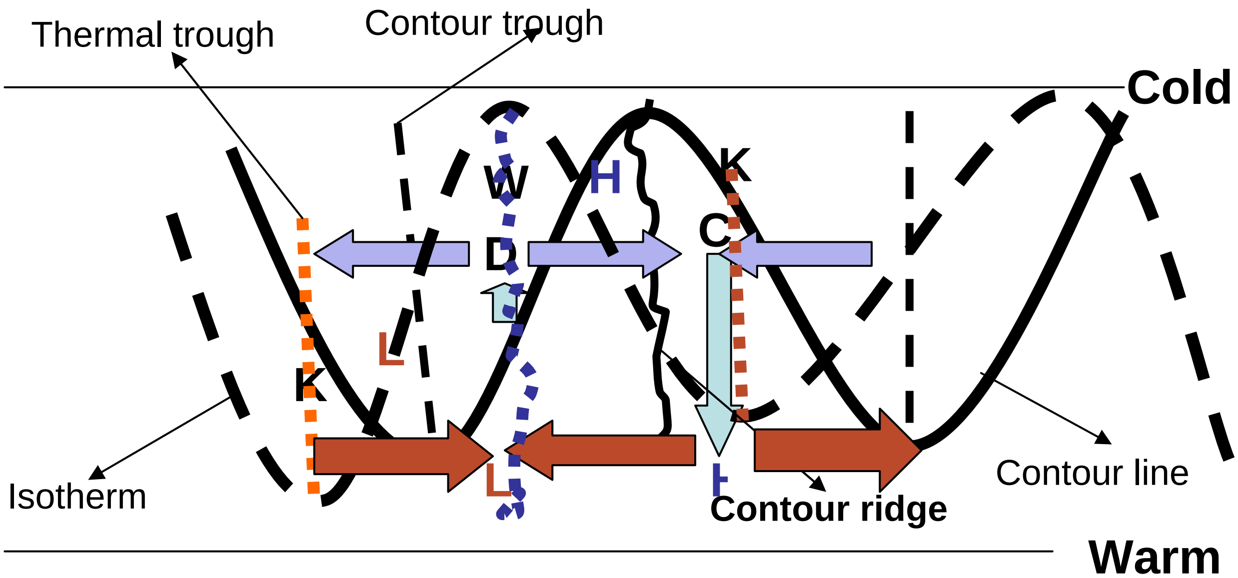

Fig.1:Baroclinic instability

- From the figure following salient features can be seen:
  - Existing N-S temperature gradient gives rise to Zonal Available potential energy (A_Z).
  - Waves in contour field gives rise to Nly cold air advection to the warmer south and Sly warm air advection to the colder north, resulting in a net reduction of A_Z.
  - Above reduction in A_Z gives rise to the generation of eddy Available potential energy (A_E), due to east-west temperature gradient, as exhibited by alternative cold (K) and warm (W) region in the wave.
  - From the figure we also see divergence ahead of contour trough and convergence ahead of contour ridge.
  - Divergence causes cooling over 'W' and convergence causes warming over 'K', resulting in a net reduction in A_E.
  - The above net reduction in A_E is attributed to the generation of eddy kinetic energy (K_E), required to drive the circulation in the vertical plane, as shown in the figure.

- To compensate the net reduction in A_E, there must be supply of cold northerly air over cold part (K) of wave and warm southerly air over warm part (W) of wave.
- The above requires that thermal trough must lag behind the contour trough. Then only a baroclinic wave grows.
- It can be shown that thermal trough should lag behind contour trough by $\pi/2$.

CISK (Conditional instability of second kind):

- This instability is a combined dynamic and thermodynamic instability.
- To understand it we consider a synoptic scale low and the atmosphere above it is already conditionally unstable.
- Due to low there will be large scale moisture convergence and as the atmosphere above the low is conditionally unstable, the moist air being positively buoyant will rise, cool and condense.
- The latent of condensation will cause divergence at upper level, which in tern will enhance low level moisture convergence.

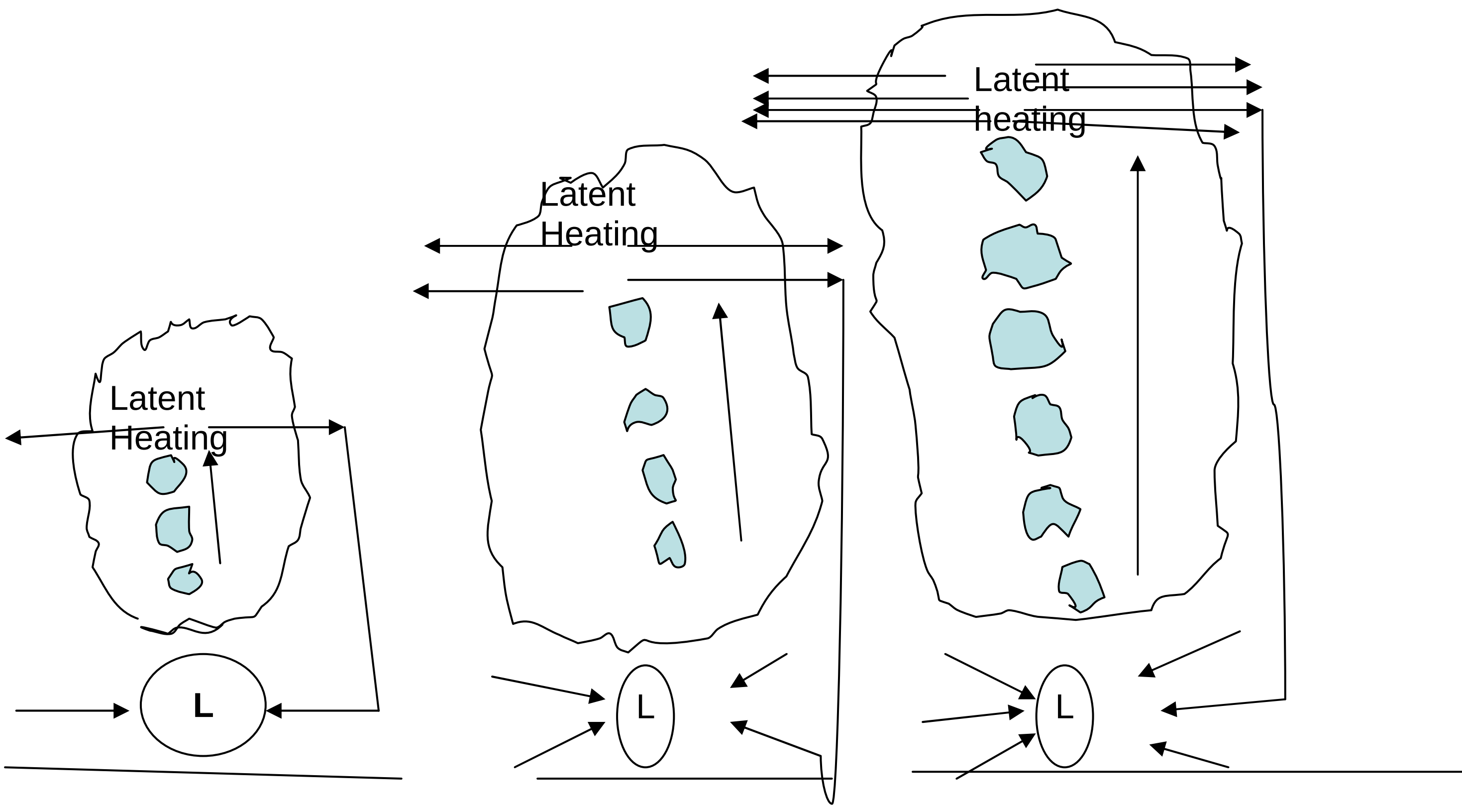

Fig2.
CISK

- The enhanced low level moisture convergence in tern will again enhance heating.
- Thus there is a co-operative mutual interaction between large scale moisture convergence and cumulus scale heating.
- The above gives rise to a different type of instability, known as CISK.
- The above has been explained schematically in fig.2.